III. Rationality of descendent series & virtual Segre/Verlinde series.

⓪ Review

$$Z^{f,g}_{S,N,\beta}(q,\underline{w}\,|\,\underline{\alpha}) := \sum_{n\in\mathbb{Z}} q^n \cdot \int_{[\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)]^{vir}} \prod_{s=1}^{N} f_s(\alpha_s^{[n]}) \cdot g(T^{vir})$$

In the last lecture, we proved that for β SW length N,

$$Z^{f,g}_{S,N,\beta}(q,\underline{w}\,|\,\underline{\alpha}) = q^{-\beta.K_S} \cdot \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \cdot \prod_{s\le t} \mathsf{C}_{s,t}^{c_1(\alpha_s).c_1(\alpha_t)} \cdot \prod_s \mathsf{D}_s^{c_2(\alpha_s)} \cdot \mathsf{E}^{\chi(\mathscr{O}_S)}$$

$$\cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \text{s.t. } \mathrm{vd}_{\beta_i}=0}} SW(\beta_1)\cdots SW(\beta_N) \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i\le j} \mathsf{W}_{i,j}^{\beta_i.\beta_j}$$

Recall that universal series contributions are defined as

$$\hat{Z}^{f,g}_{S,N,\underline{t}}(q,\underline{w}\,|\,\underline{\alpha}) = \sum_{m_1,\ldots,m_N\ge 0} q^{\sum m_i} \cdot \int_{S^{[m_1]}\times\cdots\times S^{[m_N]}} \prod_{i=1}^{N} e(\mathcal{O}^{[m_i;0]}_{\beta_i}) \cdot \frac{i^*\left(\prod_s f_s(\alpha_s^{[n]}) \cdot g(T^{vir})\right)}{e(N^{vir})}$$

$$= \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \cdot \prod_{s\le t} \mathsf{C}_{s,t}^{c_1(\alpha_s).c_1(\alpha_t)} \cdot \prod_s \mathsf{D}_s^{c_2(\alpha_s)} \cdot \mathsf{E}^{\chi(\mathscr{O}_S)} \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i\le j} \mathsf{W}_{i,j}^{\beta_i.\beta_j}$$

① Vanishing $\Rightarrow C_{s,t} = D_s = E = 1$.

$$\hat{Z}^{f,g}_{S,N,\underline{\beta}}(q, \underline{w} \,|\, \underline{\alpha}) = \sum_{m_1,\dots,m_N \geq 0} q^{\sum m_i} \int_{S^{[m_1]}\times\cdots\times S^{[m_N]}} \prod_{i=1}^{N} e\left(\mathcal{CO}^{[m_i,0]}_{\beta_i}\right) \cdot \frac{\iota^*\left(\prod_s f_s(\alpha_s^{[n]}) \cdot g(T^{vir})\right)}{e(N^{vir})}$$

$$= \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \cdot \prod_{s \leq t} \mathsf{C}_{s,t}^{c_1(\alpha_s).c_1(\alpha_t)} \cdot \prod_s \mathsf{D}_s^{c_2(\alpha_s)} \cdot \mathsf{E}^{\chi(\mathcal{O}_S)} \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i \leq j} \mathsf{W}_{i,j}^{\beta_i.\beta_j}$$

Consider a triple $(S, (\beta_1, \dots, \beta_N), (\alpha_1, \dots, \alpha_\ell))$ s.t.

$$\beta_1 = \dots = \beta_N = K_S.$$

Then we have

$$\mathcal{O}(D_i) \in \mathrm{Pic}_S^{K_S}$$

$$\mathcal{CO}^{[m_i,0]}_{\beta_i} = \left(K_S(-D_i)^{[m_i]}\right)^\vee = \left(\mathcal{O}_S^{[m_i]}\right)^\vee.$$

Claim: $m_i > 0 \Rightarrow e\left(\mathcal{CO}^{[m_i,0]}_{\beta_i}\right) = 0$

$$0 \to I_{Z_i} \to \mathcal{O}_S \to \mathcal{O}_{Z_i} \to 0$$

$$\Rightarrow 0 \to \underbrace{H^0(\mathcal{O}_S)}_{\cong\, \mathbb{C}} \to \underbrace{H^0(\mathcal{O}_{Z_i})}_{\cong\, \mathcal{O}^{[m_i]}|_{Z_i}} \to \ker\left(H^1(I_{Z_i}) \twoheadrightarrow H^1(\mathcal{O}_S)\right) \to 0$$

vector bundle.

$$\Rightarrow 0 \to \mathcal{O}_{S^{[m_i]}} \to \mathcal{O}^{[m_i]} \to \quad \to 0 \qquad \blacksquare$$

$$\hat{Z}^{\underline{f}, \underline{g}}_{S, N, \beta}(q, \underline{w} | \underline{\alpha}) = \sum_{m_1, \dots, m_N \geq 0} q^{\sum m_i} \int_{S^{[m_1]} \times \dots \times S^{[m_N]}} \prod_{i=1}^{N} e(\mathcal{O}_{\beta_i}^{[m_i ; 0]}) \cdot \frac{i^*\left(\prod_s f_s(\alpha_s^{[n]}) \cdot g(T^{\mathrm{vir}})\right)}{e(N^{\mathrm{vir}})}$$

$$= \int_{S^{[0]} \times \dots \times S^{[0]}} \frac{i^*\left(\prod_s f_s(\alpha_s^{[n]}) \cdot g(T^{\mathrm{vir}})\right)}{e(N^{\mathrm{vir}})}$$

just a point.

↪ equivariantly non-trivial.

$$= \prod_{i,s} \left(f_s(w_i)^{-r_s}\right)^{K^2} \cdot \left(f_s(w_i)\right)^{K.c_1(\alpha_s)}$$

$$= \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \cdot \prod_{s \leq t} \mathsf{C}_{s,t}^{c_1(\alpha_s).c_1(\alpha_t)} \cdot \prod_s \mathsf{D}_s^{c_2(\alpha_s)} \cdot \mathsf{E}^{\chi(\mathscr{O}_S)} \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i \leq j} \mathsf{W}_{i,j}^{\beta_i.\beta_j}$$

K_S K_S K_S^2

$$= \left(A \cdot \prod_i U_i \cdot \prod_{i \leq j} W_{ij}\right)^{K^2} \cdot \left(\prod_s B_s \cdot \prod_{i,s} V_{i,s}\right)^{K.c_1(\alpha_s)} \cdot \prod_{s \leq t} C_{s,t}^{c_1(\alpha_s).c_1(\alpha_t)} \cdot \prod_s D_s^{c_2(\alpha_s)} \cdot E^{\chi(\mathcal{O}_S)}$$

By independence of Chern numbers, we obtain

$$C_{s,t} = D_s = E = 1 \quad \blacksquare$$

② Computation of universal series by $\widetilde{K3}$.

$$\hat{Z}^{f,g}_{S,N,\beta}(q,\underline{\omega}\,|\,\underline{\alpha}) = \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_i.\beta_j}$$

Let $S := \widetilde{K3}$. Then $K_S = \mathcal{O}_S(E)$.

Given a partition $[N] = I \sqcup J$, define

$$\underline{E}_{I\sqcup J} := (\beta_1, \cdots, \beta_N) \text{ where } \begin{cases} \beta_i = E, & i \in I \\ \beta_j = 0, & j \in J \end{cases}.$$

$$\Rightarrow \hat{Z}^{f,g}_{S,N,\underline{E}_{I\sqcup J}}(q,\underline{\omega}\,|\,\underline{\alpha}) = \mathsf{A}^{-1} \cdot \prod_s \mathsf{B}_s^{E.c_1(\alpha_s)} \cdot \prod_{i\in I} \mathsf{U}_i^{-1} \cdot \prod_{i\in I,s} \mathsf{V}_{i,s}^{E.c_1(\alpha_s)} \cdot \prod_{\substack{i_1<i_2 \\ i_1,i_2\in I}} \mathsf{W}_{i_1,i_2}^{-1}$$

Enough to compute this!.

- Step 1 : Reduction to the curve geometry.
- Step 2 : Express integrand in terms of curve data.
- Step 3 : Evaluate integral by Lagrange-Bürmann formula.

Step 1 : Reduction to the curve geometry.

Recall that $CO_{\beta_i}^{[m_i,0]} = \left(K_S(-\beta_i)^{[m_i]}\right)^{\vee}$.

$$\begin{cases} i \in I \Rightarrow CO^{[m_i, 0]}_{\beta_i} = \left(\mathcal{O}^{[m_i]}\right)^\vee \\ j \in J \Rightarrow CO^{[m_j, 0]}_{\beta_j} = \left(\mathcal{O}(E)^{[m_j]}\right)^\vee \end{cases}$$

Since $e\left(\mathcal{O}^{[m_i]\vee}\right) = 0$ for $m_i > 0$, we may assume $m_i = 0 \ \forall i \in I$.

Claim : $e\left(\mathcal{O}(E)^{[m_j]\vee}\right) \cap S^{[m_j]} = (-1)^{m_j} \iota_* [E^{[m_j]}]$.

pf) Section $\theta : \mathcal{O}_S \longrightarrow \mathcal{O}_S(E)$ gives tautological section

$$\theta^{[m_j]} : \mathcal{O}_{S^{[m_j]}} \longrightarrow \mathcal{O}_S(E)^{[m_j]}.$$

At each point $Z \in S^{[m_j]}$, it is

$$\mathbb{C} = H^0(\mathcal{O}_S) \longrightarrow H^0(\mathcal{O}_S(E)) \longrightarrow H^0(\mathcal{O}_S(E)|_Z)$$

which vanishes iff $Z \subseteq E$. Vanishing locus

$$Z_{ero}\left(\theta^{[m_j]}\right) = E^{[m_j]} \overset{\iota}{\hookrightarrow} S^{[m_j]}$$

is of expected dimension $\blacksquare$

$$\hat{Z}^{f,g}_{S,N,E_{I\cup J}}(q, \underline{w} | \underline{\alpha}) = \sum_{m_j \geq 0, j \in J} (-q)^{\sum m_j} \cdot \int_{\prod_{j \in J} E^{[m_j]}} \iota^* \left(\frac{i^*\left(\prod_s f_s(\alpha_s^{[n]}) \cdot g(T^{vir})\right)}{e(N^{vir})} \right)$$

Step 2 : Express integrand in terms of curve data.

We illustrate how to do this with $\underline{\underline{N^{vir}}}$ only.

Geometry :

$$\begin{array}{ccc}
\mathrm{Quot}_S(\mathbb{C}^N, \beta, n) & \longleftarrow & T^{vir} \\
\uparrow i & & \\
\mathrm{Quot}_S(\mathbb{C}^1, \beta_1, n_1) \times \cdots \times \mathrm{Quot}_S(\mathbb{C}^1, \beta_N, n_N) & \longleftarrow & N^{vir} = (i^* T^{vir})^{mov} \\
\cong & & \\
\underbrace{S^{[m_1]} \times \cdots \times S^{[m_N]}}_{m_i = 0 \ \forall i \in I} \times \underbrace{\mathrm{Hilb}^{\beta_1} \times \cdots \times \mathrm{Hilb}^{\beta_N}}_{\mathrm{Hilb}^E = \{E\},\ \mathrm{Hilb}^0 = \{\phi\}} & & \\
\uparrow u & & \\
\prod_{j \in J} E^{[m_j]} & \longleftarrow & u^* N^{vir}
\end{array}$$

How does universal sequence $0 \to S \to \mathcal{O}^{\oplus N}_{\mathrm{Quot} \times S} \to Q \to 0$

restrict over $\prod_{j \in J} E^{[m_j]} \times S$?

$$0 \to \underbrace{\bigoplus_{i \in I} \mathcal{O}(-E)[w_i]}_{m_i = 0 \ \&\ \beta_i = E} \oplus \underbrace{\bigoplus_{j \in J} I_{Z_j}[w_j]}_{m_j \geq 0 \ \&\ \beta_j = 0} \to \bigoplus_{k=1}^{N} \mathcal{O}[w_k] \to \bigoplus_{i \in I} \mathcal{O}_E[w_i] \oplus \bigoplus_{j \in J} \mathcal{O}_{Z_j}[w_j] \to 0 .$$

(Z_j's supported on $E \subseteq S$).

$$\iota^* N^{vir} = \left(R\mathcal{H}om_p\left(\bigoplus_{i\in I} \overbrace{\mathcal{O}(-E)[w_i] \oplus \bigoplus_{j\in J} I_{Z_j}[w_j] ,\ \bigoplus_{i\in I} \mathcal{O}_E[w_i]}^{\chi(\mathcal{O}_E(E))=0} \oplus \bigoplus_{j\in J} \mathcal{O}_{Z_j}[w_j] \right)\right)^{mov}$$

$$= \sum_{i,j} R\mathcal{H}om_p(\mathcal{O}(-E), \mathcal{O}_{Z_j})[w_j - w_i] + \sum_{j,i} R\mathcal{H}om_p(I_{Z_j}, \mathcal{O}_E)[w_i - w_j]$$

$$+ \sum_{j_1 \neq j_2} R\mathcal{H}om_p(I_{Z_{j_1}}, I_{Z_{j_2}})[w_{j_2} - w_{j_1}].$$

Goal: To write these intrinsically over $E^{[m_j]}$.

Pick $Z_j \in E^{[m_j]}$. Denote $i: E \hookrightarrow S$.

e.g. $$\mathrm{Ext}^\bullet_S(I_{Z_j/S}, \mathcal{O}_E) = \mathrm{Ext}^\bullet_S(i_* I_{Z_j/E} + \cancel{\mathcal{O}(-E)}, \mathcal{O}_E)$$

$$= \mathrm{Ext}^\bullet_S(i_* I_{Z_j/E}, i_* \mathcal{O}_E)$$

$$= \mathrm{Ext}^\bullet_E(I_{Z_j/E}, \underbrace{i^! i_* \mathcal{O}_E}_{\Lambda^*_{-1} N_{E/S}})$$

$$= \mathrm{Ext}^\bullet_E(\underbrace{\mathcal{O}_E - \mathcal{O}_{Z_j}, \mathcal{O}_E - H}_{0})$$

$$= H^\bullet(\mathcal{O}_E) + H^\bullet(\mathcal{O}_{Z_j} \otimes H^2)^\vee - H^\bullet(\mathcal{O}_{Z_j} \otimes H)^\vee$$

$$= \mathbb{C} + (K_E^{[m_j]})^\vee\big|_{Z_j} - (H^{[m_j]})^\vee\big|_{Z_j} \quad \blacksquare$$

Step 3 : Evaluate integral by Lagrange–Bürmann formula.

We can also identify tautological classes $\alpha^{[m]} \in K^0(E^{[m]})$.

Since $E \simeq \mathbb{P}^1 \Rightarrow E^{[m]} \simeq \mathbb{P}^m$ with universal quotient:

$$0 \to \underbrace{\mathcal{O}_{\mathbb{P}^m}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-m)}_{= I_Z} \to \mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^1} \to \mathcal{O}_Z \to 0$$

Using this, for $\alpha \in K^0(\mathbb{P}^1)$ of rank r, deg d,

$$\alpha^{[m]} = \mathcal{O}_{\mathbb{P}^m}(-1)^{\oplus rm-d-r} - \mathcal{O}_{\mathbb{P}^m}^{\oplus d+r}.$$

$\exists$ explicit formula in terms of $\{h_j\}_{j \in J}$.

$$\widehat{Z}^{\underline{f},g}_{\underline{S},N,E_{I\sqcup J}}(q,\underline{w}\,|\,\underline{\alpha}) = \sum_{m_j \geq 0,\, j\in J} (-q)^{\sum m_j} \cdot \int_{\prod_{j\in J}\mathbb{P}^{m_j}} n^* \left(\frac{i^*\left(\prod_s f_s(\alpha_s^{[n]}) \cdot g(T^{vir})\right)}{e(N^{vir})} \right)$$

$$= \sum_{m_j \geq 0,\, j \in J} (-q)^{\sum m_j} \left[\prod_{j\in J} h_j^{m_j}\right] \prod_{j\in J} \Phi_j(h_j)^{m_j} \cdot \Psi_{I\sqcup J}(\{h_j\}_{j\in J})$$

$(d_s = E.\, c_1(\alpha_s))$

$$\Phi_j(h_j) = (-h_j)\cdot\prod_s f_s(-h_j+w_j)^{r_s}\cdot\prod_k \frac{g(-h_j+w_j-w_k)}{-h_j+w_j-w_k}$$

$$\Psi_{I\sqcup J} = \frac{1}{g(0)^{|J|}}\cdot\prod_j h_j\cdot\prod_{s,k} f_s(w_k)^{r_s+d_s}\cdot\prod_{s,j}\frac{1}{f_s(-h_j+w_j)^{r_s+d_s}}\cdot\prod_{j,k}\frac{g(h_j-w_j+w_k)}{h_j-w_j+w_k}\cdot\prod_{j_1\neq j_2}\frac{h_{j_1}-w_{j_1}-h_{j_2}+w_{j_2}}{g(h_{j_1}-w_{j_1}-h_{j_2}+w_{j_2})}$$

※ Lagrang – Bürmann formula (See e.g. [Gessel, (4.4)])

$$\sum_{m_j \geq 0,\ j \in J} (-q)^{\sum m_j} \left[\prod_{j\in J} h_j^{m_j} \right] \prod_{j \in J} \Phi_j(h_j)^{m_j} \cdot \Psi_{I \sqcup J}(\{h_j\}_{j\in J}) = \frac{\Psi_{I\sqcup J}(\{h_j\})}{K_{I \sqcup J}(\{h_j\})}$$

$$\begin{cases} K_{I\sqcup J}(\{h_j\}) = \prod_{j\in J} \left(1 - h_j \cdot \frac{\Phi_j'(h_j)}{\Phi_j(h_j)} \right) \\ -q = \frac{h_j}{\Phi_j(h_j)} \quad \text{s.t.} \quad h_j(q=0) = 0 . \end{cases}$$

Change of variables are different for each $j \in J$.

We may use another variable change $H_j := h_j - w_j$.

☆ $$q = \underbrace{\prod_s \frac{1}{f_s(-H_j)^{r_s}} \cdot \prod_{k=1}^{N} \frac{-H_j - w_k}{g(-H_j - w_k)}}_{} \quad \text{with} \quad H_j(q=0) = -w_j.$$

$F(H_j)$ with SINGLE "F".

After some easy calculation, we identify all the universal series in variables $H_1, \cdots, H_N$, $w_1, \cdots, w_N$ up to change of variables ☆.

③ Formulas for the universal series

Thm (AJLOP) If β is SW length N, then

$$Z^{f,g}_{S,N,\beta}(q, \underline{w} \,|\, \underline{\alpha}) = q^{-\beta.K} \cdot A^{K^2} \cdot \prod_s B_s^{c_1(\alpha_s).K}$$

$$\cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N \\ \mathrm{vd}_{\beta_i}=0}} SW(\beta_1)\cdots SW(\beta_N) \cdot \prod_i U_i^{\beta_i.K} \cdot \prod_{i,s} V_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i<j} W_{i,j}^{\beta_i.\beta_j} .$$

$$A = g(0)^N \cdot \prod_s \prod_{i=1}^N \frac{f_s(-H_i)^{r_s}}{f_s(w_i)^{r_s}} \cdot \prod_{i,k\in[N]} \frac{H_i+w_k}{g(H_i+w_k)} \cdot \prod_{\substack{i_1\neq i_2 \\ i_1,i_2\in[N]}} \frac{g(H_{i_1}-H_{i_2})}{H_{i_1}-H_{i_2}} \cdot \prod_{i=1}^N \left(\sum_s r_s \cdot \frac{f_s'}{f_s}(-H_i) + \sum_{k=1}^N \left(\frac{g'}{g}(-H_i-w_k) + \frac{1}{H_i+w_k} \right) \right)$$

$$U_i = \frac{1}{g(0)} \cdot \prod_s \frac{1}{f_s(-H_i)^{r_s}} \cdot \prod_{k=1}^N \frac{g(H_i+w_k)}{H_i+w_k} \cdot \prod_{\substack{i'\neq i \\ i'\in[N]}} \frac{H_{i'}-H_i}{g(H_{i'}-H_i)} \cdot \frac{H_i-H_{i'}}{g(H_i-H_{i'})} \cdot \left(\sum_s r_s \cdot \frac{f_s'}{f_s}(-H_i) + \sum_{k=1}^N \left(\frac{g'}{g}(-H_i-w_k) + \frac{1}{H_i+w_k} \right) \right)^{-1}$$

$$W_{i_1,i_2} = \frac{g(H_{i_1}-H_{i_2})}{H_{i_1}-H_{i_2}} \cdot \frac{g(H_{i_2}-H_{i_1})}{H_{i_2}-H_{i_1}} \quad / \quad B_s = \prod_{k=1}^N \frac{f_s(w_k)}{f_s(-H_k)} \quad / \quad V_{i,s} = f_s(-H_i)$$

where
$$q = \prod_s \frac{1}{f_s(-H_j)^{r_s}} \cdot \prod_{k=1}^N \frac{-H_j-w_k}{g(-H_j-w_k)} \quad \text{with} \quad H_j(q=0) = -w_j .$$

This can be specialized to the cases of interests by choosing $f_1, \dots, f_\ell, g$ appropriately.

④ Rationality of K-theoretic descendent series.

Thm (AJLOP) If β is SW length N, then

$$Z^{K}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k}) := \sum_{n\in\mathbb{Z}} q^n \chi^{\mathrm{vir}}\left(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n),\ \bigotimes_{s=1}^{\ell} \wedge^{k_s}\alpha_s^{[n]} \otimes \wedge_{-y}\Omega^{\mathrm{vir}}\right)$$

is given by a rational function in $\mathbb{Q}(y)(q)$.

(sketch of proof)

Define master K-theoretic generating series

$$Z^{\dagger}_{S,N,\beta}(q\,|\,\underline{\alpha}) := \sum_{n\in\mathbb{Z}} q^n \chi^{\mathrm{vir}}\left(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n),\ \bigotimes_{s=1}^{\ell} \wedge_{x_s}\alpha_s^{[n]} \otimes \wedge_{-y}\Omega^{\mathrm{vir}}\right).$$

$\boxed{1}$ $$Z^{K}_{S,N,\beta}(q\,|\,\underline{\alpha},\underline{k}) = \frac{1}{k_1!\cdots k_\ell!}\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_\ell}\right)^{k_\ell} Z^{\dagger}_{S,N,\beta}(q\,|\,\underline{\alpha})\Big|_{x_1=\cdots=x_\ell=0}$$

$\boxed{2}$ $$Z^{\dagger}_{S,N,\beta}(q\,|\,\underline{\alpha}) = Z^{\underline{f},g}_{S,N,\beta}(q\,|\,\underline{\alpha})$$ where

$$f_s(z) = 1+x_s e^{z},\quad g(z) = \frac{z(1-ye^{-z})}{(1-e^{-z})} \in \mathbb{Q}(x_1,\dots,x_\ell,y)[\![z]\!]$$

$\updownarrow$ $\mathrm{ch}(1+xL)$

$\updownarrow$ $\mathrm{ch}(1-yL^{\vee})\cdot \mathrm{td}(L)$

For this choice of $f_1 \cdots f_\ell, g$, multiplicative series are :

$$\mathsf{A} = (1-y)^N \prod_s \prod_{i=1}^N \frac{(1+x_s e^{-\varepsilon_s H_i})^{r_s}}{(1+x_s e^{\varepsilon_s w_i})^{r_s}} \cdot \prod_{i,k\in[N]} \frac{1-e^{-H_i-w_k}}{1-ye^{-H_i-w_k}} \cdot \prod_{\substack{i_1\neq i_2\\ i_1,i_2\in[N]}} \frac{1-ye^{-H_{i_1}+H_{i_2}}}{1-e^{-H_{i_1}+H_{i_2}}}$$

$$\cdot \prod_{i=1}^N \left(\sum_s r_s \cdot \frac{\varepsilon_s x_s e^{-\varepsilon_s H_i}}{1+x_s e^{-\varepsilon_s H_i}} + \sum_{k=1}^N \left(- \frac{e^{-H_i-w_k}+ye^{H_i+w_k}}{(1-ye^{H_i+w_k})(1-e^{-H_i-w_k})} \right) \right),$$

$$\mathsf{U}_i = \frac{1}{1-y} \prod_s \frac{1}{(1+x_s e^{-\varepsilon_s H_i})^{r_s}} \cdot \prod_{k=1}^N \frac{1-ye^{-H_i-w_k}}{1-e^{-H_i-w_k}} \cdot \prod_{\substack{i'\neq i\\ i'\in[N]}} \frac{1-e^{-H_{i'}+H_i}}{1-ye^{-H_{i'}+H_i}} \cdot \frac{1-e^{-H_i+H_{i'}}}{1-ye^{-H_i+H_{i'}}}$$

$$\boxed{\varepsilon_s = 1}.$$

$$\cdot \left(\sum_s r_s \cdot \frac{\varepsilon_s x_s e^{-\varepsilon_s H_i}}{1+x_s e^{-\varepsilon_s H_i}} + \sum_{k=1}^N \left(- \frac{e^{-H_i-w_k}+ye^{H_i+w_k}}{(1-ye^{H_i+w_k})(1-e^{-H_i-w_k})} \right) \right)^{-1},$$

$$\mathsf{W}_{i,j} = \frac{1-ye^{-H_i+H_j}}{1-e^{-H_i+H_j}} \cdot \frac{1-ye^{-H_j+H_i}}{1-e^{-H_j+H_i}},$$

$$\mathsf{B}_s = \prod_{k=1}^N \frac{1+x_s e^{\varepsilon_s w_k}}{1+x_s e^{-\varepsilon_s H_k}},$$

$$\mathsf{V}_{i,s} = 1 + x_s e^{-\varepsilon_s H_i},$$

where $$q = \prod_s \frac{1}{(1+x_s e^{-\varepsilon_s H_i})^{r_s}} \cdot \prod_{k=1}^N \frac{1-e^{H_i+w_k}}{1-ye^{H_i+w_k}} \quad \text{with} \quad H_i(q=0) = -w_i.$$

As expected for K-theoretic invariants, many exponential functions appear. ∴ We make exponential change of variables :

$$e^{H_i} = z_i \quad \& \quad e^{w_k} = t_k$$

This simplifies both universal series & change of variables.

$$\mathsf{A} = (1-y)^N \prod_s \prod_{i=1}^N \frac{(1+x_s z_i^{-\varepsilon_s})^{r_s}}{(1+x_s t_i^{\varepsilon_s})^{r_s}} \cdot \prod_{i,k\in[N]} \frac{1-z_i^{-1}t_k^{-1}}{1-yz_i^{-1}t_k^{-1}} \cdot \prod_{\substack{i_1\neq i_2\\ i_1,i_2\in[N]}} \frac{1-yz_{i_1}^{-1}z_{i_2}}{1-z_{i_1}^{-1}z_{i_2}}$$

$$\cdot \prod_{i=1}^N \left(\sum_s r_s \cdot \frac{\varepsilon_s x_s z_i^{-\varepsilon_s}}{1+x_s z_i^{-\varepsilon_s}} + \sum_{k=1}^N \Big(- \frac{z_i^{-1}t_k^{-1} + yz_i t_k}{(1-yz_i^{-1}t_k^{-1})(1-z_i^{-1}t_k^{-1})} \Big) \right),$$

$$\mathsf{U}_i = \frac{1}{1-y} \prod_s \frac{1}{(1+x_s z_i^{-\varepsilon_s})^{r_s}} \cdot \prod_{k=1}^N \frac{1-yz_i^{-1}t_k^{-1}}{1-z_i^{-1}t_k^{-1}} \cdot \prod_{\substack{i'\neq i\\ i'\in[N]}} \frac{1-z_{i'}^{-1}z_i}{1-yz_{i'}^{-1}z_i} \cdot \frac{1-z_i^{-1}z_{i'}}{1-yz_i^{-1}z_{i'}}$$

$$\cdot \left(\sum_s r_s \cdot \frac{\varepsilon_s x_s z_i^{-\varepsilon_s}}{1+x_s z_i^{-\varepsilon_s}} + \sum_{k=1}^N \Big(- \frac{z_i^{-1}t_k^{-1} + yz_i t_k}{(1-yz_i^{-1}t_k^{-1})(1-z_i^{-1}t_k^{-1})} \Big) \right)^{-1},$$

$$\mathsf{W}_{i,j} = \frac{1-yz_i^{-1}z_j}{1-z_i^{-1}z_j} \cdot \frac{1-yz_j^{-1}z_i}{1-z_j^{-1}z_i},$$

$$\mathsf{B}_s = \prod_{k=1}^N \frac{1+x_s t_k^{\varepsilon_s}}{1+x_s z_k^{-\varepsilon_s}},$$

$$\mathsf{V}_{i,s} = 1 + x_s z_i^{-\varepsilon_s},$$

where
$$q = \prod_s \frac{1}{(1+x_s z_i^{-\varepsilon_s})^{r_s}} \cdot \prod_{k=1}^N \frac{1-z_i t_k}{1-yz_i t_k} \quad \text{with} \quad z_i(q=0) = t_i^{-1}.$$

Apparently from the formula, we have some rationality

$$\mathsf{A},\ \mathsf{U}_i,\ \mathsf{W}_{i,j},\ \mathsf{B}_s,\ \mathsf{V}_{i,s} \in \mathbb{Q}(x_1,\cdots,x_\ell,y)\underbrace{(t_1,\cdots,t_N)}_{\text{symmetric}}(z_1,\cdots,z_N)$$

Note that $\forall \sigma \in S_N$,

$$U_i(z_{\sigma(1)},\cdots,z_{\sigma(N)}) = U_{\sigma(i)}(z_1,\cdots,z_N)$$

and similarly for others. This implies that

$$Z^{\dagger}_{\underline{S},N,\beta}(q,\underline{w}\,|\,\underline{\alpha}) = q^{-\beta.K_S} \cdot \mathsf{A}^{K_S^2} \cdot \prod_s \mathsf{B}_s^{K_S.c_1(\alpha_s)} \cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N\\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\underline{\beta}) \cdot \prod_i \mathsf{U}_i^{\beta_i.K_S} \cdot \prod_{i,s} \mathsf{V}_{i,s}^{\beta_i.c_1(\alpha_s)} \cdot \prod_{i<j} \mathsf{W}_{i,j}^{\beta_i.\beta_j}$$

$$\in \mathbb{Q}(x_1,\cdots,x_\ell,y)(t_1,\cdots,t_N)(z_1,\cdots,z_N)$$

which is symmetric in both $t_1,\cdots,t_N$ & $z_1,\cdots,z_N$.

Symmetricity is preserved under taking derivative, hence

$$\frac{1}{k_1! \cdots k_\ell!} \left(\frac{\partial}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_\ell}\right)^{k_\ell} \Big| _{x_1 = \cdots = x_\ell = 0} Z^{\dagger}_{S^N, \beta}(q, \underline{w} \mid \underline{\alpha}) \in \mathbb{Q}(y)(t_1, \cdots, t_N)(z_1, \cdots, z_N)$$

$S_N \circlearrowleft \quad S_N \circlearrowleft$

$$\in \mathbb{Q}(y)(t_1, \cdots, t_N)(e_1, \cdots, e_N).$$

Enough to show: $e_1, \cdots, e_N \in \mathbb{Q}(y)(t_1, \cdots, t_N)(q)$

Change of variable (after $x_1 = \cdots = x_\ell = 0$) is

$$q = \prod_{k=1}^{N} \frac{1 - z t_k}{1 - y z t_k} \longleftarrow z = z_1, \cdots, z_N \quad \text{with} \quad z_i(q=0) = t_i^{-1}.$$

$$\Updownarrow$$

$$\frac{\prod_{k=1}^{N} (1 - z t_k) - q \cdot \prod_{k=1}^{N} (1 - y z t_k)}{\left(\prod_{k=1}^{N} -t_k\right)(1 - y^N q)} = (z - z_1) \cdots (z - z_N)$$

$$= z^N - e_1 z^{N-1} + e_2 z^{N-2} \cdots + (-1)^N e_N.$$

This gives explicit formulas for

$$e_1, \cdots, e_N \in \mathbb{Q}(y)(t_1, \cdots, t_N)(q) \qquad \blacksquare$$

This procedure can be applied to give formula for Z^K for surfaces whose Seiberg-Witten theory is well-understood.

⑤ Some speculations of rationality.

Question: Does rationality come from 3-fold theory...?

Consider $X := \mathrm{Tot}(K_S) \xrightarrow{\pi} S$ & $\mathbb{C}^\times \curvearrowright X$.

Suppose that $\mathrm{Quot}_X(\mathbb{C}^N, \beta, n)$ admits $\mathbb{C}^\times$-equivariant perfect obstruction theory s.t.

$$T^{vir}_{\mathrm{Quot}_X} = R\mathcal{H}om_p(\mathcal{O}^{\oplus N}_{\mathrm{Quot}_X \times X}, \mathcal{O}^{\oplus N}_{\mathrm{Quot}_X \times X}) - R\mathcal{H}om_p(\mathcal{S}, \mathcal{S}).$$

One can show that

$$\left(\mathrm{Quot}_X(\mathbb{C}^N, \beta, n)\right)^{\mathbb{C}^\times} = \begin{array}{c}\text{nested Quot scheme}\\ \text{of } S\end{array} \supseteq \mathrm{Quot}_S(\mathbb{C}^N, \beta, n)$$

We can compute

$$\begin{tikzcd} S \arrow[r, hook, "i"] \arrow[dr, "\pi"'] & X \arrow[d, "p"] \\ & pt \end{tikzcd}$$

$$T^{vir}_{\mathrm{Quot}_X}\Big|_{\mathrm{Quot}_S} = R\mathcal{H}om_p(\mathcal{O}^N, \mathcal{O}^N) - R\mathcal{H}om_p(\overset{\mathcal{O}^N - \mathcal{Q}}{\overset{\|}{\mathcal{S}}}, \mathcal{S})$$

$$= R\mathcal{H}om_p(\mathcal{O}^N, \mathcal{Q}) + R\mathcal{H}om_p(\mathcal{Q}, \mathcal{O}^N) - R\mathcal{H}om_p(\mathcal{Q}, \mathcal{Q})$$

$$= R\mathcal{H}om_p(\mathcal{O}^N, \mathcal{Q}) - R\mathcal{H}om_p(\mathcal{O}^N, \mathcal{Q} \otimes t^{-1})^\vee - R\mathcal{H}om_p(\mathcal{Q}, \mathcal{Q})$$

$$= R\mathcal{H}om_\pi(\mathcal{O}^N, \mathcal{Q}) - R\mathcal{H}om_\pi(\mathcal{O}^N, \mathcal{Q})^\vee t - R\mathcal{H}om_\pi(\mathcal{Q}, \mathcal{Q}) + R\mathcal{H}om_\pi(\mathcal{Q}, \mathcal{Q} \otimes K_S)\, t$$

$$= R\mathcal{H}om_\pi(\mathcal{S}, \mathcal{Q}) - \left(R\mathcal{H}om_\pi(\mathcal{O}^N, \mathcal{Q}) - R\mathcal{H}om_\pi(\mathcal{Q}, \mathcal{Q})\right)^\vee t$$

$$= T^{vir}_{Quot_S} - \Omega^{vir}_{Quot_S} t$$

$\therefore$ We conclude that

$$\begin{cases} \left(T^{vir}_{Quot_X} \Big|_{Quot_S} \right)^{fix} = T^{vir}_{Quot_S} \\ N^{vir}_{Quot_S/Quot_X} = -\Omega^{vir}_{Quot_S} t \end{cases}$$

1 Homological invariants.

$$\text{``}\int_{[Quot_X]^{vir}} 1 \text{''} = \int_{[Quot_S]^{vir}} \frac{1}{e(N^{vir})} + \text{non-trivial nested contributions.}$$

$$\int_{[Quot_S]^{vir}} \frac{1}{e(N^{vir})} = \int_{[Quot_S]^{vir}} e(\Omega^{vir} \otimes t)$$

$$= \int_{[Quot_S]^{vir}} \prod_{i=1}^{vd} (t - \alpha_i) \qquad \begin{cases} t = c_1(t) \\ \Omega^{vir} \leftarrow \alpha_i : \text{Chern roots.} \end{cases}$$

$$= (-1)^{vd} \cdot e^{vir}(Quot_S).$$

2 K-theoretic invariants.

$$\text{``}\chi\left(Quot_X, \hat{\mathcal{O}}^{vir}_{Quot_X}\right)\text{''} = \chi^{vir}\left(Quot_S, \frac{K^{\frac{1}{2}}_{Quot_X}\big|_{Quot_S}}{\Lambda_{-1}(N^{vir})^\vee}\right) + \text{non-trivial nested contributions.}$$

- $$K^{vir}_{Quot_X}\Big|_{Quot_S} = \det\left(T^{vir}_{Quot_X}\Big|_{Quot_S}\right)^\vee$$

$$= \det\left(T^{vir}_{Quot_S} - \Omega^{vir}_{Quot_S} t\right)^\vee \quad \rightsquigarrow \quad K^{vir}_{Quot_S} \otimes t^{\frac{vd}{2}}$$

$\exists$ square root

$$= \left(K^{vir}_{Quot_S}\right)^{\otimes 2} t^{vd}$$

- $$\frac{1}{\Lambda_{-1}(N^{vir})^\vee} = \Lambda_{-1}(\Omega^{vir}\hbar)^\vee$$

$$= (-1)^{vd} \cdot \Lambda_{-1}(\Omega^{vir}\hbar) \otimes \det(\Omega^{vir}\hbar)^\vee$$

$$= (-1)^{vd} (K^{vir}_{\mathrm{Quot}_S})^\vee \hbar^{-vd} \otimes \Lambda_{-1}(\Omega^{vir}\hbar)$$

$$\chi^{vir}\left(\mathrm{Quot}_S, \frac{K^{\frac{1}{2}}_{\mathrm{Quot}_X}\big|_{\mathrm{Quot}_S}}{\Lambda_{-1}(N^{vir})^\vee}\right) = (-1)^{vd} \chi^{vir}\left(\mathrm{Quot}_S, \hbar^{-\frac{vd}{2}} \otimes \Lambda_{-1}(\Omega^{vir}\hbar)\right)$$

$$= (-1)^{vd} \cdot y^{-\frac{vd}{2}} \cdot \chi^{vir}(\mathrm{Quot}_S, \Lambda_{-y}\Omega^{vir})$$

$y := e^t$, $t = c_1(\hbar)$

$$= (-1)^{vd} \cdot \overline{\chi}^{vir}_{-y}(\mathrm{Quot}_S)$$

<u>Remark</u>: 1) One can put descendents for these computations.

2) Without twisting $\widehat{\Theta}^{vir}_{\mathrm{Quot}_X}$, we obtain instead

$$\chi^{vir}(\mathrm{Quot}_S, \Lambda_{-y^{-1}} T^{vir}).$$

<u>Conclusion</u>: If Quot_X admits expected virtual class, then

homological / K-theoretic descendent invariants of Quot_S. = "instanton" contributions to descendent invariants of Quot_X.

(up to normalizations)

Some hint that rationality might come from 3-fold theory:

$$\begin{cases} \widetilde{Z}^{H}_{S,N,\beta}(q) := \sum_{n\in\mathbb{Z}} q^n \int_{[\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)]^{vir}} s(T^{vir}) \\ \widetilde{Z}^{K}_{S,N,\beta}(q) := \sum_{n\in\mathbb{Z}} q^n \cdot \chi^{vir}(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n), \Lambda_{-y} T^{vir}) \end{cases}$$

are NOT rational functions in q variable in general.

Question: Can we use this speculation to conjecture more general rationality?

e.g. Elliptic genera...

$*$ $\mathrm{Quot}_S(\mathbb{C}^N,\beta,n) \longleftrightarrow \mathrm{Pair}_S(\mathbb{C}^N,\beta,n)$

$$\mathrm{Quot}_S(\mathbb{C}^N,\beta,n) \ni [\mathcal{O}_S^{\oplus N} \xrightarrow{f} Q] \qquad \mathrm{Pair}_S(\mathbb{C}^N,\beta,n) \ni [\mathcal{O}_S^{\oplus N} \xrightarrow{f} Q]$$

- Q arbitrary
- f surjective

- Q pure of $\dim = 1$
- $\mathrm{coker}(f)$ zero dimension.

Question: Does descendent series of Pair spaces of surfaces also yield rationality?

⑥ Virtual Segre/Verlinde correspondence.

Motivated from the study of (non-virtual) Segre/Verlinde series for $S^{[n]}$, we define

$$\begin{cases} S_{S,N,\beta}(q|\alpha) := \sum_{n\in\mathbb{Z}} q^n \cdot \int_{[\mathrm{Quot}_S(\mathbb{C}^N,\beta,n)]^{vir}} s(\alpha^{[n]}) \\ V_{S,N,\beta}(q|\alpha) := \sum_{n\in\mathbb{Z}} q^n \cdot \chi^{vir}(\mathrm{Quot}_S(\mathbb{C}^N,\beta,n), \det\alpha^{[n]}) \end{cases}$$

They corresponds to $Z^{f,g}_{S,N,\beta}(q|\alpha)$ for choices of

Segre $\leftarrow$ $f(z) = \frac{1}{1+z}$, $g(z) = 1$

Verlinde $\leftarrow$ $f(z) = e^z$, $g(z) = \frac{z}{1-e^{-z}}$.

Suppose that β : SW length N.

$$\mathsf{S}_{S,N,\beta}(q,\underline{w}|\alpha) = q^{-\beta.K_S}\cdot \mathsf{A}^{K_S^2}\cdot \mathsf{B}^{K_S.c_1(\alpha)}\cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N\\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\underline{\beta})\cdot \prod_i \mathsf{U}_i^{\beta_i.K_S}\cdot \prod_i \mathsf{V}_i^{\beta_i.c_1(\alpha)}\cdot \prod_{i<j}\mathsf{W}_{i,j}^{\beta_i.\beta_j}$$

$$\mathsf{V}_{S,N,\beta}(\widetilde{q},\underline{\widetilde{w}}|\alpha) = \widetilde{q}^{-\beta.K_S}\cdot \widetilde{\mathsf{A}}^{K_S^2}\cdot \widetilde{\mathsf{B}}^{K_S.c_1(\alpha)}\cdot \sum_{\substack{\beta=\beta_1+\cdots+\beta_N\\ \mathrm{vd}_{\beta_i}=0}} \mathrm{SW}(\underline{\beta})\cdot \prod_i \widetilde{\mathsf{U}}_i^{\beta_i.K_S}\cdot \prod_i \widetilde{\mathsf{V}}_i^{\beta_i.c_1(\alpha)}\cdot \prod_{i<j}\widetilde{\mathsf{W}}_{i,j}^{\beta_i.\beta_j}$$

To state the correspondence, let

$$e^{-\widetilde{w}_k} = 1+w_k, \quad (-1)^N q = \prod_{k=1}^N (1+w_k)\cdot \widetilde{q}.$$

Thm (AJLOP)

$$\mathsf{A} = \widetilde{\mathsf{A}},$$
$$\mathsf{B} = \widetilde{\mathsf{B}},$$
$$\mathsf{V}_i = \widetilde{\mathsf{V}}_i,$$
$$\mathsf{W}_{i,j} = \widetilde{\mathsf{W}}_{i,j} \cdot \mathsf{V}_i \cdot \mathsf{V}_j,$$
$$\mathsf{U}_i = (-1)^{N-1} \cdot \prod_{k=1}^{N} (1 + w_k) \cdot \mathsf{B} \cdot \mathsf{V}_i \cdot \widetilde{\mathsf{U}}_i$$

e.g. If $\beta = 0$, then only A, B series are involved.

$$\therefore \quad S_{s,N,0}(q \mid \alpha) = V_{s,N,0}((-1)^N q, \alpha).$$

For the punctual Quot scheme ($\beta = 0$), $V_{s,N}(q, \alpha)$ exhibits symmetry exchanging $N \longleftrightarrow \mathrm{rk}(\alpha)$.

Define $\quad \mu(\alpha) := \dfrac{K \cdot c_1(\alpha)}{\mathrm{rk}(\alpha)}$.

Thm (AJLOP) • $\alpha, \tilde{\alpha}$: K-theory classes of rank $r, N \geq 1$

• $\mu(\alpha) = \mu(\tilde{\alpha})$.

$$\Rightarrow \quad V_{s,N}(q \mid \alpha) = V_{s,r}(q \mid \tilde{\alpha}),$$ in other words,

$$\chi^{vir}(\mathrm{Quot}_S(\mathbb{C}^N, n), \det \alpha^{[n]}) = \chi^{vir}(\mathrm{Quot}_S(\mathbb{C}^r, n), \det \tilde{\alpha}^{[n]})$$

This reminds the level-rank duality (strange duality) for moduli of stable bundles of curves.

Thank You